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On Families of Sets Represented in Theories

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ABSTRACT: A necessary and sufficient condition is given for a family of sets to be the family of all sets representable in a theory.

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The purpose of this paper is to answer the following questions: (1) Let F be a family of sets[1]. What are necessary and sufficient conditions that F be the family of all sets represented in some consistent standard theory[2]? (2) What are necessary and sufficient conditions that F be the family of all sets represented in some consistent <u>axiomatizable</u> standard theory?[3]

I shall prove:

THEOREM 1. <u>F is the family of all sets represented in some consistent standard theory if and only if F is closed under intersection, finite addition and subtraction[4], and contains the null set and the "universal" set (i.e. the set Nn of all non-negative integers).</u>

THEOREM 2. <u>F is the family of all sets represented in some consistent axiomatizable standard theory if and only if F is a recursively enumerable family of recursively enumerable sets[5]; F contains the null set and the "universal" set; and F is closed under intersection and finite addition and subtraction.</u>

As an example of a consequence of THEOREM 1, we may cite the fact that, since the Π_1 sets (the sets whose complements are recursively enumerable) satisfy the closure conditions mentioned in the theorem, there exists a theory T with the property that <u>all</u> and only Π_1 sets are represented in T. Similarly, it follows from THEOREM 2 that, since the recursive sets satisfy the conditions given (that they form a recursively enumerable family in the sense on n.5 was first proved by Dekker), there exists an <u>axiomatizable</u> theory T with the property that <u>all and only</u> recursive sets are represented in T. This result has been previously obtained by Shoenfield (in a stronger form); and the result about Π_1 sets can likewise be obtained by a quite different construction than the one used here. However, it is of interest to see these results not as isolated curiosities, but as special cases of very general theorems.

1. General Remarks. $\{F_1, F_2, \ldots\}$ will be a family of sets which contains the null set, the "universal" set Nn, and is closed under intersection and finite addition and subtraction; $P_1, P_2, \ldots$ will be an infinite list of monadic predicate letters; $\bar{n}$ will be the nth formal integer; T will be the theory whose axioms are $\bar{n} \neq \bar{m}$ for each pair n,m such that $n \neq m$; $P_i(\bar{n})$ for each i,n such that $n \in F_i$; and $(x)(P_{i_1}(x) \& \ldots \& P_{i_K}(x) . v . P_{j1}(x) \& \ldots \& P_{j_N}(x))$ for each pair $\{P_{i1}, \ldots, P_{iK}\}$, $\{P_{j1}, \ldots, P_{j_N}\}$ of disjoint finite sets $(K \geq 1,\ N \geq 1)$ of predicate letters from the list $P_1, P_2, \ldots$; $A_1, A_2, \ldots A_n \vdash B$ will be used to mean (where $n \geq 0$) that there is a proof of B from assumptions $A_1, A_2, \ldots A_n$ in first order predicate calculus with identity; and $\vdash_T B$ will mean that B is a theorem (valid sentence) of T.

2. Proofs. To prove Theorems 1 and 2 we need the following lemmas:

LEMMA 1: Let $\{F_1, F_2, \ldots\}$ be the family of all sets represented in some consistent standard theory S. Then $F_1, F_2, \ldots$ is closed under intersection, finite addition, and finite subtraction, and contains the null set and the universal set.

Proof: Closure under intersection is obvious, since if the w.f.f.

(well formed formula) $A(x)$ represents F_i and $B(x)$ represents F_j, then $A(x) \& B(x)$ represents $F_i \cap F_j$. The null set is represented by any self contradictory w.f.f with one free variable; the universal set is represented by any valid w.f.f. with one free variable; and finally the sets $F_i \cup \{n_1, n_2, \ldots, n_k\}$ and $F_i - \{n_1, n_2, \ldots, n_k\}$ are represented by the formulas $A(x) \vee x = \overline{n}_1 \vee \ldots \vee x = \overline{n}_k$ and $A(x) \& x \neq \overline{n}_1 \& \ldots \& x \neq \overline{n}_k$ respectively.

LEMMA 2: <u>P_i represents F_i in</u> T.

Proof: If $n \in F_i$, then $P_i(\overline{n})$ is an axiom of T, and hence $\vdash_T P_i(\overline{n})$. Now suppose $n \tilde{\in} F_i$, and consider the following interpretation of T: for all m, $\overline{m}$ designates n; P_j is assigned the universal set as extension for $j \neq i$, and P_i is assigned as its extension the set $Nn - \{n\}$. This interpretation is a true interpretation of T, and according to it the sentence $P_i(\overline{n})$ is false. Hence $P_i(\overline{n})$ is not a theorem of T.

LEMMA 3. <u>If $\vdash_T P_{i_1}(x) \& \ldots \& P_{i_M}(x) \supset A(x)$, where $M \geq 0$[6] and $A(x)$ is a w.f.f. with one free variable, then $A(x)$ represents one of the F_i in</u> T.

Proof: (By course-of-values induction on M.) Suppose $M = 0$. Then $\vdash_T A(x)$; hence $A(x)$ represents Nn.

Suppose the lemma holds for $M < N$, and let $\vdash_T F_{i_1}(x) \& \ldots \& P_{i_N}(x) \supset A(x)$. Let $\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_k$ be all of the formal integers that occur in $A(x)$. If $A(\overline{t})$ is never provable unless $\vdash_T P_{i_1}(\overline{t}) \& \ldots \& P_{i_N}(\overline{t})$ or $t \in \{s_1, s_2, \ldots, s_k\}$, then $A(x)$ represents a set that can be obtained from $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_N}$ by finite addition, and hence one of the F_i. Now suppose that $\vdash_T A(\bar{t})$, where it is not the case that $\vdash_T P_{i_1}(\bar{t})\&\ldots\&P_{i_N}(\bar{t})$, and $t \neq s_k$. Let the following be all of the axioms needed for some proof of $A(\bar{t})$ in T:

$P_{j_1}(\bar{t}),\ldots,P_{j_U}(\bar{t});\ \bar{t} \neq \bar{n}_1,\ldots,\bar{t} \neq \bar{n}_j;\ A_1,\ldots,A_S;$ where the A_i are all of the axioms not containing $\bar{t}$ used in the proof. If $U = 0$, then, by the Deduction Theorem, $A_1,\ldots,A_S \vdash \bar{t} \neq \bar{n}_1\&\ldots\&t \neq \bar{n}_j \supset A(\bar{t})$; hence, since t does not occur in $A_1,A_2,\ldots,A_S$, and t is not one of the s_i, $A_1,A_2,\ldots,A_S \vdash (x)(x \neq \bar{n}_1\&\ldots\&x \neq n_j \supset A(x))$. Then $\vdash_T (x)(x \neq \bar{n}_1\&\ldots\&x \neq \bar{n}_j \supset A(x))$, and $A(x)$ represents $Nn - W$, where W has to be a subset of $\{n_1,\ldots,n_j\}$, and hence finite. On the other hand, if $U \neq 0$, then by a similar argument $A_1,A_2,\ldots,A_S \vdash (x)(P_{j_1}(x)\&\ldots\&P_{j_U}(x) \supset (x \neq \bar{n}_1\&\ldots\&x \neq \bar{n}_j \supset A(x)))$, and so $\vdash_T P_{j_1}(x)\&\ldots\&P_{j_U}(x) \supset (x \neq \bar{n}_1\&\ldots\&x \neq \bar{n}_j \supset A(x))$. But we assumed $\vdash_T P_{i_1}(x)\&\ldots\&P_{i_N}(x) \supset A(x)$, so

(1) $\vdash_T (P_{i_1}(x)\&\ldots\&P_{i_N}(x)\ .v.\ P_{j_1}(x)\&\ldots\&P_{j_U}(x)) \supset (x \neq \bar{n}_1\& \ldots\&x \neq \bar{n}_j \supset A(x))$.

If the P_i's and the P_j's are all distinct, then $\vdash_T (x)(F_{i_1}(x)\&\ldots\&P_{i_N}(x)\ .v.\ P_{j_1}(x)\&\ldots\&P_{j_N}(x))$, and hence $\vdash_T (x \neq \bar{n}_1\&\ldots\&x \neq \bar{n}_j \supset A(x))$ and $A(x)$ represents $Nn - w$, where W is a finite set. And if the P_i's and the P_j's are not all distinct, then $P_{i_1}(x)\&\ldots P_{i_N}(x).v.P_{j_1}(x)\&\ldots\&P_{j_U}(x)$ is quantificationally equivalent (in fact, equivalent by propositional calculus) to $F_{k_1}(x)\&\ldots\&P_{k_H}(x)\ \&\ (P_{i_r}(x)\&P_{i_{r'}}(x)\&\ldots\&P_{i_{r(D)}}(x)$ $.v.\ P_{j_s}(x)\&P_{j_{s'}}(x)\&\ldots\&P_{j_{s(Q)}}(x))$, where $P_{k_1},P_{k_2},\ldots,P_{k_H}$ are all of the P's that occur both among the P_i and among the P_j, while $P_{i_r}, P_{i_{r'}}, P_{i_{r''}}, \dots, P_{i_r(D)}$ are the P_i that do not also occur among the P_j, and similarly $P_{j_s}, P_{j_{s'}}, P_{j_{s''}}, \dots, P_{j_s(Q)}$ are the P_j that do not also occur among the P_i. Moreover, D cannot = 0 (otherwise $\vdash_T P_{i_1}(\bar{t}) \& \dots \& P_{i_N}(\bar{t})$, contrary to the choice of t), and we may assume that $Q \neq 0$ (since otherwise we would have $U < N$, and the lemma would follow by the induction hypothesis[7]). Thus $(x)(P_{i_r}(x) \& \dots \& P_{i_r(D)}(x) \ .v.\ P_{j_s}(x) \& \dots \& P_{j_s(Q)}(x))$ is an axiom of T, so that $P_{i_1}(x) \& \dots \& P_{i_N}(x) \ .v.\ P_{j_1}(x) \& \dots \& P_{j_U}(x)$ is provably equivalent to $P_{k_1}(x) \& \dots \& P_{k_H}(x)$, where $H < N$. Hence the lemma follows by the induction hypothesis and the fact that since (1) is a theorem of T, $\vdash_T P_{k_1}(x) \& \dots \& P_{k_H}(x) \supset (x \neq \bar{n}_1 \& \dots \& x \neq \bar{n}_j \supset A(x))$.

LEMMA 4. <u>The family of all sets represented in an axiomatizable theory is a recursively enumerable family of recursively enumerable sets.</u>

Proof: Let the w.f.fs of S (where S is any axiomatizable theory) with one free variable be effectively listed as $A_1(x), A_2(x), \dots$. The predicate $P(i,n) =_{df} \vdash_S A_i(\bar{n})$ is a recursively enumerable predicate (to verify this, assuming Church's Thesis, note that it can be written in the form (Ex)Prf(x,n,i), where Prf(x,n,i) is the decidable, and hence recursive, predicate "x is the godel number of a proof of the formula that results when $\bar{n}$ is put for all occurences of 'x' in $A_i(x)$". Moreover, $A_i(x)$ represents $\{n \mid P(i,n)\}$, or $\{f(i)\}$ (cf. n. 5), where[8] $f(i) = S_1^1(e,i)$ and e is a gödel number of P.

Proof of THEOREM 1. By LEMMA 1, we have "only if". To prove "if" (i.e., to show that the conditions given in the theorem are sufficient) we shall show that if $\{F_1, F_2, \ldots\}$ satisfies the conditions, then $\{F_1, F_2, \ldots\}$ is the family of all sets represented in T (where T is the theory mentioned in §1).

By LEMMA 2, F_i is represented in T (for i = 1,2,...). So it suffices to show that for every w.f.f. A(x) of T, A(x) represents one of the F_i. Accordingly, let A(x) be a w.f.f. of T with x as its only free variable, and let $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_k$ be all the formal integers that occur in A(x). If $\vdash_T A(\bar{t})$ only when $t \in \{s_1, s_2, \ldots, s_k\}$, then A(x) represents a finite set, and hence one of the F_i (noting that all finite sets can be obtained from the null set by finite addition). Now suppose $\vdash_T A(\bar{t})$ where $t \not\in \{s_1, s_2, \ldots, s_k\}$. Let the following be all of the axioms needed for some proof of $A(\bar{t})$ in T: $P_{i_1}(\bar{t}), \ldots, P_{i_M}(\bar{t})$; $\bar{t} \neq n_1, \ldots, \bar{t} \neq \bar{n}_j$; $A_1, \ldots, A_S$; where the A_k are all of the axioms not containing $\bar{t}$ used in the proof. Then $A_1, A_2, \ldots, A_S \vdash P_{i_1}(\bar{t}) \& \ldots \& P_{i_M}(\bar{t}) \supset (\bar{t} \neq \bar{n}_1 \& \ldots \& \bar{t} \neq \bar{n}_j \supset A(\bar{t}))$; hence $A_1, A_2, \ldots, A_S \vdash P_{i_1}(x) \& \ldots \& P_{i_M}(x) \supset (x = \bar{n}_1 \& \ldots \& x = \bar{n}_j \supset A(x))$; and hence $\vdash_T P_{i_1}(x) \& \ldots \& P_{i_M}(x) \supset (x \neq \bar{n}_1 \& \ldots \& x = \bar{n}_j \supset A(x))$. Then by LEMMA 3, $x \neq n_1 \& \ldots \& x \neq n_j \supset A(x)$ represents one of the F_i, and hence A(x) represents one of the F_i (cf. n.7).

Proof of THEOREM 2. The proof is similar to the proof of THEOREM 1, except that LEMMA 4 must also be used for the "only if" part of the theorem, and we must note that what we have given for this case is a recursively enumerable set of axioms. The axiomatizability of T (in the sense of recursive axiomatizability) then follows by Craig's Theorem.

FOOTNOTES

1) Terminology: In this paper "set" means set of non-negative integers, except when there is indication to the contrary. A formula $P(x)$ (with one free variable x) is said to "represent" a set S in a theory T if for all integers n, $n \in S$ if and only if $P(\bar{n})$ is a theorem of T (N.B. it is _not_ required that $P(\bar{n})$ should be _refutable_ in T --- i.e., that $\sim P(\bar{n})$ should be provable in T --- when $n \tilde{\in} S$). The term "represent" comes from _Undecidable Theories_. ($n \tilde{\in} S$ is an abbreviation for $\sim n \in S$.)

2) By a "standard theory" I mean a "theory is standard formalization" in the sense in which that term is used in _Undecidable Theories_, in which there are terms (called _formal integers_ in the sequel), say $\bar{0}$, $\bar{1}$, $\bar{2}$, ... (which may be interpreted as designating 0, 1, 2, ...) such that $\bar{n} \neq \bar{m}$ is provable for all n,m such that $n \neq m$.

3) A theory in standard formalization is called "axiomatizable" in _Undecidable Theories_ if the set of valid sentences in identical with the set of first-order consequences of some _recursive_ subset (called the set of "axioms"). (Instead of "recursive" it would be better to say "solvable", in the sense of Post, since strictly speaking the recursiveness of a set of formulas depends upon the godel numbering employed, whereas "solvability" is defined directly for sets of expressions in any finite alphabet.)

4) A set B will be said to come from a set A by _finite addition_ (resp. _finite subtraction_) if $B = A \cup W$ (resp. $A - W$) where W is a finite set.

5) Following Kleene, let $\{n\}$ be the nth partial recursive function in the standard enumeration. (This notation is not to be confused with the notation $\{n| \ldots\}$, for the set of all n satisfying the condition ..., nor with the notation $\{A_1, A_2, A_3, \ldots\}$, for the set consisting of $A_1, A_2, A_3, \ldots$.) We shall identify each partial recursive function with its domain, for the purpose of enumerating the recursively enumerable sets: thus $\{n\}$ will alternatively be thought of, where convenient, as "the nth recursively enumerable set, in the standard enumeration." A family F is called a "recursively enumerable family of recursively enumerable sets" if the members of F are $\{t(0)\}$, $\{t(1)\}$, ... , for some general recursive function t.

6) If M = 0, the $\supset$ is to be understood as deleted.

7) More precisely, it would follow from the induction hypothesis that $x \neq \overline{n}_1 \& \ldots x \neq \overline{n}_j \supset A(x)$ represents one of the F_i. But $x \neq n_1 \& \ldots \& x \neq n_j$ $A(x)$ represents a superset with at most finitely many more members than the set represented by A(x) (as is clear from the fact that this formula can also be written $x = n_1 \vee x = \overline{n}_2 \vee \ldots \vee x = \overline{n}_j \vee A(x)$ can be obtained from this F_i by finite subtraction. Hence A(x) also represents one of the F_i (since the F_i are closed under finite subtraction).

8) $S^1_1(e,i)$ is a primitive recursive function whose value for any e,i) is a Gödel number of $\{x|P_e(i,x)\}$, where P_e is the eth 2-place recursively enumerable predicate in the standard enumeration. This function is constructed in <u>Introduction to Metamathematics.</u>

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